USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 3 - Year 12 - Academic Year 2000-2001

Gene A. Berg, Editor

1/3/12. Find the smallest positive integer with the property that it has divisors ending with every decimal digit; i.e., divisors ending in 0, 1, 2, ..., 9

Solution 1 by Rishi Gupta (8/CA): Let us first look at the prime factors of the answer. There must be a 5, because any number ending in 5 is divisible by 5. There must also be a 2, for the same reason. Therefore, so far, we have factors: $2 \times 5 = 10$, and the numbers 0, 1, 2, and 5 are covered.

We have the numbers 3, 4, 6, 7, 8, and 9 left.

If we can find multiples of the three odd numbers (3, 7, and 9), then the even numbers are covered because of the multiple of 2 ($2 \times 3 = 6$, $2 \times 7 = 14$, $2 \times 9 = 18$). Therefore all we need to worry about is 3, 7, and 9.

One solution (which may not be smallest) would be to use the LCM of 3, 7, and 9, which is 63. So the answer would have to be less than or equal to 63×10.

Let's see if any of the numbers ending in 7 have factors ending in 3 and 9. Seven and 17 are primes, but 27 has 3 and 9 as factors. So now $27 \times 10 = 270$ is a smaller solution. Any other possible solution below 27 would have to use a multiple of 7 or 17. The only candidates are 14 and 21, neither of which has a factor of 9 or 19.

Verification: 270 is divisible by 1<u>0</u>, <u>1</u>, <u>2</u>, <u>3</u>, 5<u>4</u>, <u>5</u>, <u>6</u>, 2<u>7</u>, 1<u>8</u>, and <u>9</u>.

Therefore the solution is $27 \times 10 =$ **270.**

Solution 2 by Matthew Pelc (12/PA): Solution **270.**
Call the desired integer N. All positive integers have 1 as divisor, so it is immediately dealt with. The only way a divisor of N can end in zero is for it to be a multiple of 10, thus 2 and 5 must be prime factors of N, and N is a multiple of 10. Divisors ending in 5 would have to be multiples of 5, but we already have taken care of 5. So 2×5 covers 0, 1, 2, and 5.

Any multiplication with 5 just produces another number ending in 0 or 5, so 5 is not really a consideration in determining the remaining needed factors. Any divisor ending with 7 automatically gives us another ending with 4 (because 2 is already a factor of N), and similarly any ending with 9 gives us another factor ending with 8. The smallest and most economical solution for 7 and 9 would be $27 = 3^3 = 3 \times 9$. The 4 ($27 \times 2 = 54$) and 8 ($9 \times 2 = 18$) are taken care of simultaneously. Notice 3 and 6 are covered since 3 is a factor and $6 = 2 \times 3$. So $2 \times 5 \times 27 = 270$ is the smallest positive integer with divisors ending in each decimal digit. If we are not sure, we can check each multiple of 10 up to 270; but notice only 70, 140, 170 and 210 have divisors ending with 7. Of these only 210 has a divisor ending in 3, but 210 has no divisor ending in 8. 27 is the best because it contains 3 and 9, and with 2 it takes care of 3, 4, 6, 7, and 9 all at once.

Solution N = **270**.

Solution 3 by Mike Church (12/PA): The smallest such integer is **270**.
Proof: First, since the desired integer has a divisor ending in 0, it is clear that this integer must be divisible by 10. Hence we can eliminate all integers not divisible by 10.

That our integer is divisible by an integer ending in 7 implies it is divisible by an integer among the set {7, 17, ...}. Hence, we can remove all other integers from our consideration. Thus the five smallest candidates for our desired integer are:

70, 140, 170, 210, 270

Next, it is desired that our integer have a factor ending with 3, so it must be divisible by one among 3, 13, 23, and so on. Thus we can eliminate 70, 140, and 170, and hence there remain

210 and 270

The divisors of 210 are 1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, and 210. As this set lacks elements whose last digits are 8 or 9, 210 can also be disqualified.

Hence, all integers less than 270 have been disqualified. Now evaluate the divisors of 270:

1, 2, 3, 5, 6, 9, 10, 15, 18, 27, 30, 45, 54, 90, 135, and 270.

On inspection, 270 indeed does have the desired property.

Editor's Comment: We are grateful to Professor Bruce Reznick of the University of Illinois for this nice problem, which he first published (via Martin Gardner) in "Science Fiction Puzzle Tales" in the early 1980's.

2/3/12. Assume that the irreducible fractions between 0 and 1, with denominators at most 99, are listed in ascending order. Determine which two fractions are adjacent to $\frac{17}{76}$ in this listing.

Editor's comment: For a discussion of *continued fractions* and *Farey series* see the Editor's Comments following the solutions.

Solution 1 by Christopher Lyons (12/CA):

We consider the continued fraction expansion of $\frac{17}{76}$:

$$\frac{17}{76} = 0 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{8}}} = [0;4, 2, 8]$$

To find the fraction that is directly below $17/76$, we must realize that changing the 8 in the expansion to a higher number would, in fact, make the overall number smaller. So we must figure out how much to add to the 8 in order to turn the overall fraction into one directly below it on the list. Let us write the number *below* $17/76$ as having the continued fraction expansion

$$x = \left[0;4, 2, 8 + \frac{1}{a}\right],$$

where a is a positive real number. When we condense x into its common fractional form, we have

$$x = \frac{17a + 2}{76a + 9}.$$

We know the denominator is less than or equal to 99, so $a \leq \frac{90}{76}$. We also know that both $17a$ and $76a$ must be positive integers. But since 17 and 76 are coprime, a must be an integer itself. The only positive integer less than $90/76$ is 1, so $a = 1$, and

$$x = \frac{19}{85}.$$

To find the fraction directly *above* $17/76$, we must *subtract* some amount from the 8. We call this larger fraction *y*, and write its continued fraction expansion as

$$y = \left[0;4, 2, 8 - \frac{1}{b}\right],$$

where b is a positive real number. When we condense y into its traditional rational form, we find

$$y = \frac{17b - 2}{76b - 9}.$$

Again, we know the denominator cannot exceed 99, so $b \leq \frac{108}{76}$. Once more, b must be an integer due to the lack of common factors between 17 and 76. Therefore, $b = 1$, and

$$y = \frac{15}{67}.$$

So

$$\frac{19}{85} < \frac{17}{76} < \frac{15}{67}.$$

Solution 2 by Lisa Leung (10/MD): The irreducible fractions between 0 and 1 with denominators at most 99, listed in ascending order, describes a Farey series of order 99.

Two basic theorems that describe characteristics of successive terms in a Farey series F_n of order n are:

(1) If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms of the Farey series, then $kh' - hk' = 1$.

(2) If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms of the Farey series, then $k + k' > n$.

By using theorem (1) with h/k set to $17/76$ where h is 17 and k is 76,

$$76h' - 17k' = 1.$$

This is similar to solving $76h' \equiv 1 (\bmod 17)$. This is solved when $h' \equiv 15 (\bmod 17)$. Substituting 15 into the first equation yields $k' = 67$, and $h'/k' = 15/67$.
Thus

$$\frac{17}{76} < \frac{15}{67}.$$

Next, by using theorem (1), but with h'/k' as $17/76$,

$$17k - 76h = 1.$$

This is similar to solving $17k \equiv 1 (\bmod 76)$. This is satisfied when $k \equiv 9 (\bmod 76)$. However, when $k = 9$, it contradicts theorem (2). Thus $k \neq 9$. When $k = 85$, it satisfies both theorems. When substituted into the equation, $\frac{h}{k} = \frac{19}{85}$.

Thus, $\frac{19}{85} < \frac{17}{76} < \frac{15}{67}$.

Solution 3 by Jason Chiu (12/WY): **Answer**: $\frac{19}{85} < \frac{17}{76} < \frac{15}{67}$.

Theorem. If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms of the Farey series F_n, then $kh' - hk' = 1$.

For several proofs of this well-known theorem, see G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Fifth Edition, Oxford University Press, London (1979).

Let $\frac{a}{b}$ and $\frac{a'}{b'}$ respectively denote the fractions left-adjacent and right-adjacent to $17/76$ in F_{99}. By the contrapositive of the Theorem, $17b - 76a = 1$ and $17b' - 76a' = -1$. To solve the first Diophantine equation, apply Euclid's Algorithm to obtain

$$76 = 4 \cdot 17 + 8,$$

$$17 = 2 \cdot 8 + 1,$$
$$8 = 1 \cdot 8 + 0.$$

Backing up through these equations gives

$$1 = 17 - 2 \cdot 8 = 17 - 2(76 - 4 \cdot 17) = 9 \cdot 17 - 2 \cdot 76,$$

yielding the solution $(a, b) = (2, 9)$ to the equation $17b - 76a = 1$. An alternate way to obtain this solution is to compute the convergents to the continued fraction expansion of $\frac{17}{76}$. The convergents in this expansion are

$$\frac{P_0}{Q_0} = 0, \quad \frac{P_1}{Q_1} = 0 + \frac{1}{4} = \frac{1}{4}, \quad \frac{P_2}{Q_2} = 0 + \frac{1}{4 + \frac{1}{2}} = \frac{2}{9}, \quad \frac{P_3}{Q_3} = 0 + \frac{1}{4 + \frac{1}{2 + \frac{1}{8}}} = \frac{17}{76}$$

$\frac{P_2}{Q_2} = \frac{2}{9}$ also gives the solution $(a, b) = (2, 9)$ to the equation $17b - 76a = 1$.

Since gcd(17,76) = 1, the general solution of the equation $17b - 76a = 1$ is $a = 2 + 17t$, $b = 9 + 76t$. For any solution (a, b) to $17b - 76a = 1$,

$$k = \frac{17}{76} - \frac{a}{b} = \frac{1}{76b}$$

so that choosing the largest value of *b* minimizes *k*. Hence the ordered pair $(a, b) = (19, 85)$ gives the smallest value of *k* and $19/85$ is left adjacent to $17/76$.

Similarly, the general solution to $17b' - 76a' = -1$ is $a' = -2 + 17t$, $b' = -9 + 76t$ so the ordered pair $(a', b') = (15, 67)$ gives the smallest value

$$k' = \frac{a'}{b'} - \frac{17}{76} = \frac{1}{76b'}$$

and $15/67$ is right-adjacent to $17/76$.

Editor's Comment: A similar problem appeared in the January 1999 issue of *Mathematical Digest*, an excellent South African mathematical journal for high school students. This problem does allow for computer solutions; we welcome them, but commend such solutions only if they are based on a clever algorithm and deal with accuracy. We thank our problem editor, Dr. George Berzsenyi, for this problem.

This interesting problem gives us an opportunity to expand on two interesting mathematical topics, continued fractions and Farey series. *Continued fractions* were discussed in the *Solutions to Round 1 of Year 10*, available on the USAMTS web site.

By a *Farey series* F_n of order *n*, we mean the set of all fractions $\frac{h}{k}$ with $0 \le h \le k$, gcd*(h,k)* = 1, $1 \le k \le n$, and arranged in ascending order of magnitude. For example, F_5 is

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

These series have remarkable properties, some of which are mentioned in the solutions above. I restate two of the theorems here so you might easily confirm them for this example:

(1) If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms of the Farey series, then $kh' - hk' = 1$.

(2) If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms of the Farey series, then $k + k' > n$.

Proofs of these and other properties are given in G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Fifth Edition, Oxford University Press, London (1979).

3/3/12. Let $p(x) = x^5 + x^2 + 1$ have roots r_1, r_2, r_3, r_4, r_5. Let $q(x) = x^2 - 2$. Determine the product $q(r_1)q(r_2)q(r_3)q(r_4)q(r_5)$.

Solution 1 by Sofia Leibman (8/OH): We can write the polynomial $x^5 + x^2 + 1$ in the form $(x - r_1)(x - r_2)\ldots(x - r_5)$.

The product $q(r_1)q(r_2)\ldots q(r_5) = (r_1^2 - 2)(r_2^2 - 2)\ldots(r_5^2 - 2)$.

$r_i^2 - 2 = (\sqrt{2} - r_i)(-\sqrt{2} - r_i)$.

So the product can be written as

$$\begin{aligned}
&(\sqrt{2} - r_1)(\sqrt{2} - r_2)\ldots(\sqrt{2} - r_5)(-\sqrt{2} - r_1)(-\sqrt{2} - r_2)\ldots(-\sqrt{2} - r_5)\\
&= p(\sqrt{2})p(-\sqrt{2})\\
&= [(\sqrt{2})^5 + (\sqrt{2})^2 + 1] \cdot [(-\sqrt{2})^5 + (-\sqrt{2})^2 + 1]\\
&= -23
\end{aligned}$$

The solution is -23.

Solution 2 by Sarah Emerson (12/WA):

Any polynomial of degree n with leading coefficient 1 can be factored as

$$(x - z_1)(x - z_2)\ldots(x - z_{n-1})(x - z_n)$$

where $z_1, z_2, \ldots z_n$ are the roots of the polynomial, are complex numbers of the form $a + bi$, (a and b are real numbers and either a or b, or both, can be zero).

Therefore, $p(x) = x^5 + x^2 + 1$ can be expressed as

$$p(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)(x - r_5).$$

Also, $q(x) = x^2 - 2$ has two roots, s_1 and s_2, so $q(x) = (x - s_1)(x - s_2)$.

Then $q(r_1) = (r_1 - s_1)(r_1 - s_2)$ and

$q(r_1)q(r_2)q(r_3)q(r_4)q(r_5) =$

$= (r_1 - s_1)(r_1 - s_2)(r_2 - s_1)(r_2 - s_2)(r_3 - s_1)(r_3 - s_2)(r_4 - s_1)(r_4 - s_2)(r_5 - s_1)(r_5 - s_2)$

$= (s_1 - r_1)(s_1 - r_2)(s_1 - r_3)(s_1 - r_4)(s_1 - r_5)(s_2 - r_1)(s_2 - r_2)(s_2 - r_3)(s_2 - r_4)(s_2 - r_5)$

since we changed an even number of signs and reordered terms. This is equivalent to $p(s_1)p(s_2)$.

Factoring $q(x) = x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ gives the two roots of *q(x)*: $s_1 = \sqrt{2}$ and $s_2 = -\sqrt{2}$.

Plug these roots of *q(x)* into the original equation for *p(x)*.

$$q(r_1)q(r_2)q(r_3)q(r_4)q(r_5) = p(\sqrt{2})p(-\sqrt{2})$$
$$= [(\sqrt{2})^5 + (\sqrt{2})^2 + 1] \cdot [(-\sqrt{2})^5 + (-\sqrt{2})^2 + 1]$$
$$= -23 .$$

This method will work with any two polynomials; however, if both polynomials are of orders that are odd numbers, the product of $f(\text{roots of } g(x))$ will be the negative of $g(\text{roots of } f(x))$ because the odd number of negative signs will not all cancel out.

Editor's Comment: We are thankful to Professor Rob Hochberg of the University of Connecticut for this interesting problem.

4/3/12. Assume that each member of the sequence $\langle \Diamond_i \rangle_{i=1}^{\infty}$ is either a **+** or a **–** sign. Determine the appropriate sequence of **+** and **–** signs so that

$$2 = \sqrt{6 \Diamond_1 \sqrt{6 \Diamond_2 \sqrt{6 \Diamond_3 \ldots}}} .$$

Also determine what sequence of signs is necessary if the sixes in the nested roots are replaced by sevens. List all integers that work in the place of sixes and the sequences of signs that are needed with them.

Solution by Ho Seung (Paul) Ryu (9/KS):

- Firstly, we notice that $2 = \sqrt{6-2}$. So, if we replace the 2 on the right side with the identical expression $\sqrt{6-2}$ in infinite number of times, we will have obtained $2 = \sqrt{6 - \sqrt{6 - \sqrt{6 - \sqrt{6 - \ldots}}}}$. So the sequence of signs for 6 is simply -,-,-,-,... .
- For number 7, $2 = \sqrt{7-3}$, which is of little help, but $3 = \sqrt{7+2}$, so $2 = \sqrt{7 - \sqrt{7+2}}$. So, by replacing 2 on the right side, we get $2 = \sqrt{7 - \sqrt{7 + \sqrt{7 - \sqrt{7 + \ldots}}}}$, just alternating

signs -,+,-,+, ...

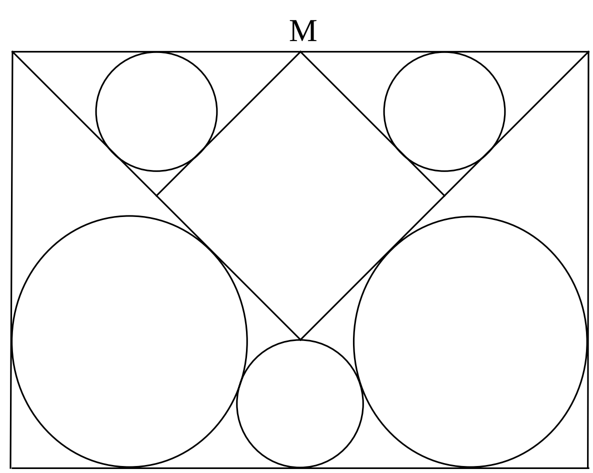
- For the number 8, we will prove that 8 cannot be used. That is for a simple reason, that $2 = \sqrt{8-4}$, but no sequence of radicals and 8s can be larger than $\sqrt{8+\sqrt{8+\ldots}}$. This quantity is given by $x = \sqrt{8+\sqrt{8+\ldots}} = \sqrt{8+x}$. $x^2 = 8+x$ and thus $x^2 - x - 8 = 0$. $x = \dfrac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-8)}}{2} = \dfrac{1 \pm \sqrt{33}}{2}$. This is less than 4, so 8 does not work in the problem.
- No numbers greater than 8 satisfy the conditions set forth in the previous statement, so now we can investigate numbers less than 6.
- Now for 5, $2 = \sqrt{5-1}$, $1 = \sqrt{5-4}$, $4 = \sqrt{5+11}$, $11 = \sqrt{5+116}$. The numbers just never cease to increase, so we can never have a working sequence of signs.
- For 4, $2 = \sqrt{4-0}$, and we are stuck right away.
- For 3, $2 = \sqrt{3+1}$, $1 = \sqrt{3-2}$, and so $2 = \sqrt{3+\sqrt{3-\sqrt{3+\sqrt{3-\ldots}}}}$. Thus the sequence of signs is +,-,+,-, ...
- As for 2, $2 = \sqrt{2+2}$, so $2 = \sqrt{2+\sqrt{2+\ldots}}$ and we have sequence +,+,+,+, ...
- Lastly, for 1, $2 = \sqrt{1+3}$, $3 = \sqrt{1+8}$, $8 = \sqrt{1+63}$, and we just keep increasing. Therefor, 1 does not work.

Thus for the number six, the required sequence of signs is just a string of minus signs -,-,-,-, ... The numbers that work in place of six, and the sequence of signs needed are:

2: All plus signs +,+,+,+, ...
3: Alternating plus/minus +,-,+,-, ...
7: Alternating minus/plus -,+,-,+, ...

Editor’s comment: This wonderful problem was proposed by Dr. Rodrigo Gomez of NSA.

5/3/12. Three isosceles right triangles are erected from the larger side of a rectangle into the interior of the rectangle, as shown on the right, where M is the midpoint of that side. Five circles are inscribed tangent to some of the sides and to one another as shown. One of the circles touches the vertex of the largest triangle.



Find the ratios among the radii of the five circles.

Solution by Lisa Leung (10/MD): Since the triangles are isosceles right triangles that are erected from the larger side of a rectangle, the two circles marked as *a* are congruent and the two circles marked as *c* are congruent.

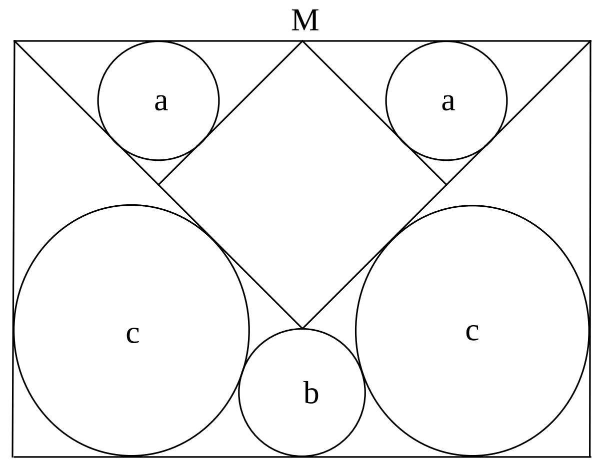


Figure 1.

Without loss of generality, let the length of the rectangle be 1 unit, and r_a, r_b, and r_c be the radius of circle *a*, *b* and *c* respectively.

As shown in Figure 2,

$$r_a + r_a\sqrt{2} = \frac{1}{4}$$

$$r_a = \frac{1}{4(1+\sqrt{2})} = \frac{\sqrt{2}-1}{4} \qquad (1)$$

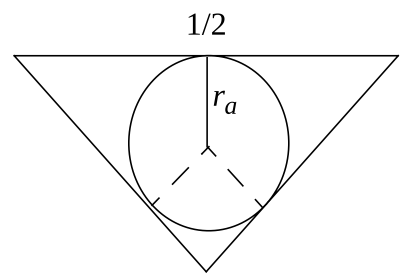


Figure 2.

From the base of figure 3,

$$\frac{1}{2} = \sqrt{(r_c+r_b)^2-(r_c-r_b)^2} = 2\sqrt{r_b r_c}+r_c \qquad (2)$$

From the width of the rectangle in Figure 3,

$$(1+\sqrt{2})r_c = \frac{1}{\sqrt{2}}\left(\frac{1}{2}+2r_b\right)$$

$$r_b = \left(1+\frac{\sqrt{2}}{2}\right)r_c - \frac{1}{4} \qquad (3)$$

Substitute into (2),

$$\frac{1}{2} = 2\sqrt{\left(1+\frac{\sqrt{2}}{2}\right)r_c^2-\frac{r_c}{4}}+r_c$$

$$\frac{1}{2}-r_c = 2\sqrt{\left(1+\frac{\sqrt{2}}{2}\right)r_c^2-\frac{r_c}{4}}$$

$$(1-2r_c)^2 = 16\times\left[\left(1+\frac{\sqrt{2}}{2}\right)r_c^2-\frac{r_c}{4}\right]$$

$$4r_c^2-4r_c+1 = (16+8\sqrt{2})r_c^2-4r_c$$

$$r_c^2 = \frac{1}{4(3+2\sqrt{2})} = \frac{1}{4(1+\sqrt{2})^2}$$

Take the positive square root,

$$r_c = \frac{1}{2(1+\sqrt{2})} = \frac{\sqrt{2}-1}{2} = 2r_a \qquad (4)$$

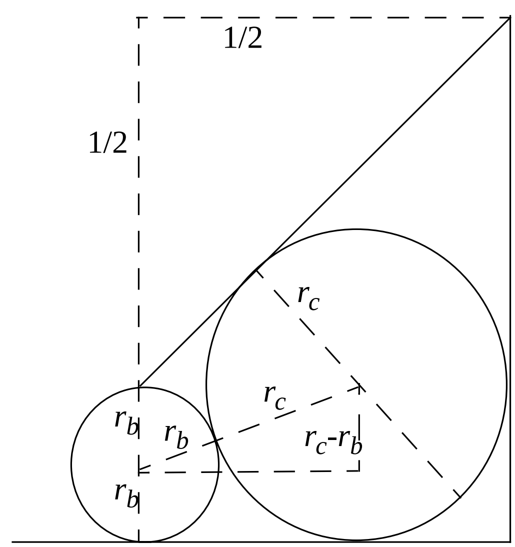


Figure 3.

Substitute into (3),

$$r_b = \left(1 + \frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}-1}{2}\right) - \frac{1}{4} = \frac{\sqrt{2}-1}{4} = r_a$$

Thus, the three smallest circles have the same radii. The ratios among the radii of the five circles are:

$$\boldsymbol{a{:}a{:}b{:}c{:}c = 1{:}1{:}1{:}2{:}2}.$$

Editor's comments: We are indebted to Professor Hiroshi Okumura of Japan for this wonderful shungaku problem. Professor Okumura is a longtime enthusiastic promoter of the Japanese equivalent of the USAMTS.